Oscillatory multiband dynamics of free particles: The ubiquity of *zitterbewegung* effects

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(Dated: February 4, 2008)

In the Dirac theory for the motion of free relativistic electrons, highly oscillatory components appear in the time evolution of physical observables such as position, velocity, and spin angular momentum. This effect is known as *zitterbewegung*. We present a theoretical analysis of rather different Hamiltonians with gapped and/or spin-split energy spectrum (including the Rashba, Luttinger, and Kane Hamiltonians) that exhibit analogs of *zitterbewegung* as a common feature. We find that the amplitude of oscillations of the Heisenberg velocity operator $\mathbf{v}(t)$ generally equals the uncertainty for a simultaneous measurement of two linearly independent components of $\mathbf{v}$. It is also shown that many features of *zitterbewegung* are shared by the simple and well-known Landau Hamiltonian describing the dynamics of two-dimensional (2D) electron systems in the presence of a magnetic field perpendicular to the plane. Finally, we also discuss the oscillatory dynamics of 2D electrons arising from the interplay of Rashba spin splitting and a perpendicular magnetic field.

PACS numbers: 73.21.-b, 71.70.Ej, 03.65.Pm

I. INTRODUCTION AND OVERVIEW

The Dirac equation[1,2,3] was derived to obtain a relativistic generalization of Schrödinger's approach to quantum physics that describes the dynamics of single-electron quantum states. While it served as an important stepping stone towards a more complete description of quantum-electrodynamic effects, Dirac theory has occasionally been regarded with some suspicion. In particular, the effect of *zitterbewegung*[4] showed that solutions of the Dirac equation exhibit peculiarities that are inconsistent with classical intuition in a more fundamental way than nonrelativistic quantum physics. The *zitterbewegung* is an oscillatory dynamics of observables induced by the Dirac equation, with a frequency of the order of $2mc^2/\hbar$, where m is the electron mass, c is the speed of light and $\hbar$ is the Planck constant. The amplitude of oscillations in a particle's position is of the order of the Compton wave length. Subsequently, *zitterbewegung* attracted some interest as a possible way to understand the intrinsic magnetic moment of the electron.[5,6] Later, on the level of fundamental physics, the advent of quantum field theory obviated the need to discuss relativistic quantum theory in terms of a first-quantized, Schrödinger-type theory. Present interest in the Dirac equation ranges from hadronic physics[7] over lattice gauge theory[8] to recent efforts[9] to incorporate relativistic effects into quantum-chemistry calculations.

A Dirac-like dynamics causing analogs of *zitterbewegung* was also predicted for electrons moving in crystalline solids,[10,11] in particular for narrow-gap semiconductors,[12] carbon nanotubes,[13] graphene sheets,[14] tunnel-coupled electron-hole bilayers[15] and superconductors.[16] All these systems are characterized by having the relevant electron excitations grouped into two bands separated by a nonzero energy gap so that their energy spectrum is similar to the spectrum of the Dirac Hamiltonian. A recent study[17] of two-dimensional (2D) electron systems in inversion-asymmetric semiconductor heterostructures showed the presence of an oscillatory motion analogous to *zitterbewegung* arising from spin splitting of the energy levels. The spin splitting corresponds to an energy gap that vanishes for momentum $p \to 0$. A similar situation occurs for electronic excitations in the bulk of an ideal graphene sheet.[14]

These findings indicate the need to understand *zitterbewegung*-like effects on a more general level. In Ref. 18, the authors presented a general formula for the Heisenberg position operator $\mathbf{r}(t)$ in systems that can be described by effective 2×2 Hamiltonians.[19] In the present work, we have investigated the oscillatory dynamics of Heisenberg observables such as position $\mathbf{r}(t)$, velocity $\mathbf{v}(t) = d\mathbf{r}/dt$, orbital angular momentum $\mathbf{L}(t)$, and spin $\mathbf{S}(t)$ in a variety of qualitatively different models that describe the motion of free (quasi-)particles. Besides the Dirac Hamiltonian, we have studied three Hamiltoni-

ans frequently used in semiconductor physics to describe the dynamics of (quasi-free) Bloch electrons in the vicinity of the fundamental gap, the Rashba,[20] Luttinger,[21] and Kane[22] Hamiltonians. A number of striking features emerge quite generally in all these models, thus illustrating remarkable similarities between time evolutions generated by rather different Hamiltonians. We suggest that these common features can be used to extend the concept of *zitterbewegung* to a broader class of quantum Hamiltonians for free (quasi-)particles. Our analysis shows that this generalized notion of *zitterbewegung* is manifested, in addition to the oscillatory unitary time evolution of observables, also by uncertainty relations characterizing the measurement of such observables. These two aspects turn out to be closely related. In particular, they can be described, for each of the models considered here, by the same set of parameters. Also, we identify the typical scales (lengths, velocities, and frequencies) that characterize *zitterbewegung*-like oscillatory motion. We emphasize that this extended notion of *zitterbewegung* is entirely based on quantum mechanical concepts. In an alternative, semiclassical approach one would identify a *zitterbewegung* relative to a suitable classical dynamics as, e.g., in Ref. 23. In some cases the conclusions will be different from those obtained within the present approach. The most general aspects of our study can be summarized as follows:

(i) An oscillatory motion occurs in the time evolution of free (quasi-) particles when the energy spectrum of the corresponding Hamiltonian H is characterized by one or several energy gaps. Besides the Dirac model, an important example are Bloch electrons in solids,[10,11] whose quantum dynamics are described by effective free-particle Hamiltonians that incorporate the effect of the periodic lattice potential.

(ii) In the case of two-band models (e.g., the Dirac, Rashba, and Luttinger models), *zitterbewegung*-like effects are generally characterized by an amplitude operator $\mathbf{F}$ and a frequency operator $\hat{\omega}(\mathbf{p})$. These two quantities enter the expression for the velocity operator in the Heisenberg picture, which can be decomposed as $\mathbf{v}(t) = \bar{\mathbf{v}}(t) + \tilde{\mathbf{v}}(t)$, where the mean part is

$$\bar{\mathbf{v}}(t) = \frac{\partial H}{\partial \mathbf{p}} - \mathbf{F} \tag{1a}$$

and the oscillating part is

$$\tilde{\mathbf{v}}(t) = \mathbf{F}\, \mathrm{e}^{-i\hat{\omega}(\mathbf{p})\,t} = \mathrm{e}^{i\hat{\omega}(\mathbf{p})\,t}\, \mathbf{F} \quad . \tag{1b}$$

Here, $\hbar\hat{\omega}(\mathbf{p})$ is related to the energy difference between states having the same momentum $\mathbf{p}$, but belonging to different subspaces (i.e., energy bands) of the Hamiltonian. The operator $\mathbf{F}$, which anticommutes with $\hat{\omega}(\mathbf{p})$, determines the magnitude of oscillations in the velocity components but also enters the expression for the mean part. We can integrate Eq. (1) to get the Heisenberg position operator that can be decomposed in the same way, $\mathbf{r}(t) = \bar{\mathbf{r}}(t) + \tilde{\mathbf{r}}(t)$, where

$$\bar{\mathbf{r}}(t) \;=\; \mathbf{r} + \bar{\mathbf{v}}t + \mathbf{F}\,\frac{1}{i\hat{\omega}(\mathbf{p})} \quad , \tag{2a}$$

$$\tilde{\mathbf{r}}(t) \;=\; -\mathbf{F}\,\frac{\mathrm{e}^{-i\hat{\omega}(\mathbf{p})t}}{i\hat{\omega}(\mathbf{p})} \quad . \tag{2b}$$

Similarly, we get the time derivative of $\mathbf{v}(t)$,

$$\dot{\mathbf{v}}(t) = i\hat{\omega}(\mathbf{p})\,\mathbf{F}\;\mathrm{e}^{-i\hat{\omega}(\mathbf{p})t} \quad . \tag{3}$$

The operators $\mathbf{F}$ and $\hat{\omega}(\mathbf{p})$ govern also the oscillations in the Heisenberg time evolution of the orbital angular momentum operator $\mathbf{L}(t)$, and the spin operators $\mathbf{S}(t)$. In systems with more than two bands (Kane and Landau-Rashba models), more than one characteristic frequency and amplitude operator can appear.

(iii) For each model describing an oscillatory multiband dynamics of free particles, the components of the velocity operator $\mathbf{v}(t)$ do not commute. This can be written as an uncertainty relation that takes the form (apart from a prefactor of order one)

$$\Delta v_j\, \Delta v_k \geq \tilde{v}^2 \qquad (j \neq k), \tag{4}$$

where $\tilde{v}$ is the amplitude of the oscillatory motion, see Eq. (1b). The uncertainty relations (4) are an integral part of our analysis.[24]

(iv) The velocity operator $\mathbf{v}(t)$ does not commute with the Hamiltonian. Although we discuss the motion of free (quasi-)particles, the components of $\mathbf{v}(t)$ are not constants of the motion, see Eq. (3). On the other hand, momentum $\mathbf{p}$ is always a constant of the motion. This implies that none of the models discussed here provides a simple relation between momentum $\mathbf{p}$ and velocity $\mathbf{v}$.

(v) The counterintuitive properties of $\mathbf{r}(t)$ and $\mathbf{v}(t)$ arise because $\mathbf{r}(t)$ mixes different subspaces $\mathcal{H}_j$ that are associated with the different bands in the energy spectrum of H. Thus we can interpret *zitterbewegung*-like phenomena as an interference effect. In the case of two-band models, one can replace $\mathbf{r}$ by the part $\bar{\mathbf{r}}$ that leaves the subspaces $\mathcal{H}_\pm$ associated with the '+' and '−' bands separately invariant,

$$\bar{\mathbf{r}} = P_+\,\mathbf{r}\,P_+ + P_-\,\mathbf{r}\,P_- \quad , \tag{5}$$

where $P_\pm$ are projection operators onto these subspaces. The result coincides with the mean part $\bar{\mathbf{r}}(t)$ of $\mathbf{r}(t)$ introduced in Eq. (2a), i.e., *zitterbewegung*-like effects are removed by the projection (5). This result can be understood from a different perspective by analyzing the amplitude operator $\mathbf{F}$. We get

$$\mathbf{F}P_+ = P_-\mathbf{F} \quad \text{and} \quad \mathbf{F}P_- = P_+\mathbf{F}\,, \tag{6}$$

i.e., $\mathbf{F}$ maps states associated with the '+' band onto states associated with the '−' band and vice versa. An alternative definition of $\bar{\mathbf{r}}(t)$ is obtained by applying the inverse unitary transformation to $\mathbf{r}(t)$ that makes H diagonal. The same techniques can also be applied to $\mathbf{v}(t)$

to obtain $\bar{\mathbf{v}}(t)$ given in Eq. (1a). The components of $\bar{\mathbf{v}}(t)$ commute; hence they can be measured simultaneously [unlike Eq. (4)]. They also commute with the Hamiltonian so that they are constants of the motion.

(vi) In every case considered, *zitterbewegung*-like phenomena are manifested also by oscillations of the orbital angular momentum $\mathbf{L}(t)$ and spin $\mathbf{S}(t)$. At the same time, the *total* angular momentum $\mathbf{J}$ does *not* oscillate as a function of time. As expected for a model of a free particle, $\mathbf{J}$ is a constant of the motion, i.e., it commutes with the Hamiltonian. From a different perspective, this implies that the oscillations of $\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}$ and $\mathbf{S}(t)$ must cancel each other, which is possible only if the oscillations of $\mathbf{r}(t)$ and $\mathbf{S}(t)$ have a common origin. For the Rashba Hamiltonian, the oscillatory motion of $S_z(t)$ corresponds to the well-known and experimentally observed[25] spin precession in the effective magnetic field of the Rashba term.

The following Sections II–V are devoted to a detailed discussion of *zitterbewegung* effects arising in systems whose time evolution is governed by the Dirac,[3] Rashba,[20] Luttinger,[21] and Kane[22] Hamiltonians. Remarkable formal similarities between the oscillatory behavior of observables in these models are established, as outlined above. Next we show in Sec. VI that the familiar Landau model of 2D electrons subject to a perpendicular magnetic field[26] exhibits essentially all the features attributed to the extended notion of *zitterbewegung* in previous sections. We finish our case studies in Sec. VII by investigating the quantum oscillatory dynamics of 2D electrons arising from the interplay of Rashba spin splitting and a perpendicular magnetic field.[27] Conclusions and a summary of open questions are presented in Sec. VIII. For easy reference, we provide a number of relevant basic formulae in the Appendix.

II. DIRAC HAMILTONIAN

Our discussion of *zitterbewegung* for the Dirac Hamiltonian follows, for the most part, Ref. 3. We include this section with an overview of the effect's salient features to provide a reference frame and notation for our following discussion of solid-state analogies.

The Dirac Hamiltonian H_D describes a free relativistic electron or positron. It can be written in the form

$$H_D = c\,\boldsymbol{\alpha}\cdot\mathbf{p} + \beta\, mc^2 \quad , \tag{7}$$

where

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} , \qquad \beta = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} , \tag{8}$$

and $\boldsymbol{\sigma}$ is the vector of Pauli spin matrices. (Here we assume magnetic field $B = 0$. See Ref. 28 for the generaliziation to finite B.) The energy eigenvalues of H_D are $E_\pm(\mathbf{p}) = \pm\Lambda_D$, where

$$\Lambda_D = \sqrt{m^2c^4 + c^2p^2}. \tag{9}$$

Note that $H_D^2 = \Lambda_D^2$. In the Schrödinger picture, the velocity operator reads

$$\mathbf{v} = \frac{i}{\hbar}[H_D, \mathbf{r}] = \frac{\partial H_D}{\partial \mathbf{p}} = c\,\boldsymbol{\alpha} \quad , \tag{10}$$

so that the components of $\mathbf{v}$ have the two discrete eigenvalues $\pm c$. In the Heisenberg picture, we get Eq. (1) with

$$\mathbf{F} = \frac{\partial H_D}{\partial \mathbf{p}} - \frac{c^2\,\mathbf{p}}{H_D} , \qquad \hat{\omega}(\mathbf{p}) = \frac{2H_D}{\hbar} \quad . \tag{11}$$

The operator $\mathbf{F}$ mediates a coupling between states with positive and negative energies; see below. The oscillatory part $\tilde{\mathbf{v}}(t)$ of $\mathbf{v}(t)$, given in Eq. (1b), describes the *zitterbewegung*. The frequency of the *zitterbewegung* is (at least) of the order of $\omega_D \equiv 2mc^2/\hbar$. Integrating $\mathbf{v}(t)$ yields the position operator $\mathbf{r}(t)$ in the Heisenberg picture, see Eq. (2), which contains again the quickly oscillating term $\mathrm{e}^{-i\hat{\omega}(\mathbf{p})t}$. The oscillatory time dependence is similar to the motion of a nonrelativistic particle in the presence of a magnetic field [see Eq. (48) and discussion in Sec. VI]. An illuminating discussion of *zitterbewegung* based on a numerical calculation of the time evolution of wave packets can be found in Ref. 29.

It turns out[3] that orbital angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and spin $\mathbf{S}$, which is defined as

$$\mathbf{S} = -\frac{i\hbar}{4}\,\boldsymbol{\alpha}\times\boldsymbol{\alpha} = \frac{\hbar}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} , \tag{12}$$

show the phenomenon of *zitterbewegung*, too. For the orbital angular momentum, we have

$$\mathbf{L}(t) = \mathbf{r}(t)\times\mathbf{p} = \mathbf{L} + \mathbf{F}\times\mathbf{p}\,\frac{1-\mathrm{e}^{-i\hat{\omega}(\mathbf{p})t}}{i\hat{\omega}(\mathbf{p})} . \tag{13a}$$

The time evolution of spin in the Heisenberg picture reads

$$\mathbf{S}(t) = \mathbf{S} - \mathbf{F}\times\mathbf{p}\,\frac{1-\mathrm{e}^{-i\hat{\omega}(\mathbf{p})t}}{i\hat{\omega}(\mathbf{p})} . \tag{13b}$$

Thus it follows from Eqs. (13a) and (13b) that the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ does not oscillate as a function of time,

$$\mathbf{J}(t) = \mathbf{J} = \mathbf{L} + \mathbf{S} \quad , \tag{13c}$$

which reflects the fact that $[\mathbf{J}, H_D] = 0$.

We can estimate the magnitude of *zitterbewegung* by evaluating the square of $\tilde{\mathbf{v}}(t)$ (Ref. 4). This yields

$$\tilde{v}^2(t) = \frac{c^2\,(2\Lambda_D^2 + m^2c^4)}{\Lambda_D^2} , \tag{14}$$

i.e., $\tilde{v}^2$ varies between $3c^2$ in the nonrelativistic limit and $2c^2$ in the relativistic limit. (Note that, although $\tilde{v}^2 > c^2$, no measurable velocity exceeds c.) On the other hand, the components of the velocity operator $\mathbf{v}$ do not commute. Equations (10) and (12) imply that

$$[v_j, v_k] = \frac{4ic^2}{\hbar}\,\varepsilon_{jkl}\,S_l \quad . \tag{15a}$$

Diagonalizing this equation yields the uncertainty relation for $j \neq k$

$$\Delta v_j \, \Delta v_k \geq c^2 = \left(\tfrac{1}{2}\, \omega_D \lambda\!\!\!^{-}_D\right)^2 , \tag{15b}$$

where $\lambda\!\!\!^{-}_D = \hbar/(mc)$ is the Compton wave length. Thus both the magnitude and the uncertainty of the *zitterbewegung* are given by c^2 (Ref. 24). We can also estimate the spatial amplitude of the *zitterbewegung* using the decomposition $\mathbf{r}(t) = \bar{\mathbf{r}}(t) + \tilde{\mathbf{r}}(t)$, see Eq. (2). We get for the oscillating part

$$\tilde{r}^2(t) = \frac{\hbar^2 c^2 \left(2\Lambda_D^2 + m^2 c^4\right)}{4\Lambda_D^4}, \tag{16}$$

i.e., in the nonrelativistic limit, the amplitude of *zitterbewegung* is approximately $\lambda\!\!\!^{-}_D$, and it is given by the de Broglie wave length $\lambda\!\!\!^{-}_B = \hbar/p$ in the relativistic limit.

It is well-known[3] that *zitterbewegung* is caused by a coupling between the states with positive energies ("particles", subspace $\mathcal{H}_+$) and negative energies ("antiparticles", subspace $\mathcal{H}_-$). Thus one can eliminate the oscillations of $\mathbf{r}(t)$ by projecting $\mathbf{r}$ on $\mathcal{H}_\pm$ as in Eq. (5) and the result coincides with Eq. (2a). The components $\bar{r}_j$ of $\bar{\mathbf{r}}$ do not commute:

$$[\bar{r}_j, \bar{r}_k] = -i \frac{\hbar c^2}{\Lambda_D^2}\, \varepsilon_{jkl} \bar{S}_l \quad , \tag{17a}$$

where

$$\bar{\mathbf{S}} = \mathbf{S} - \mathbf{F} \times \mathbf{p}\, \frac{\hbar}{\hat{\omega}(\mathbf{p})} \tag{17b}$$

is the spin operator analogous to Eq. (1a) that does not mix the subspaces of positive and negative energy states. Diagonalizing Eq. (17a) yields the uncertainty relation for $j \neq k$

$$\Delta \bar{r}_j \, \Delta \bar{r}_k \geq \frac{\hbar^2 c^2}{4} \, \frac{\sqrt{m^2 c^4 + |\varepsilon_{jkl}|\, c^2 p_l^2}}{\Lambda_D^3} \quad . \tag{17c}$$

Thus we obtain in the nonrelativistic limit ("$v \ll c$")

$$\Delta \bar{r}_j \, \Delta \bar{r}_k \gtrsim \tfrac{1}{4}\, \lambda\!\!\!^{-}_D{}^2 \quad . \tag{17d}$$

In the opposite (ultrarelativistic) limit ("$v \lesssim c$") we have

$$\Delta \bar{r}_j \, \Delta \bar{r}_k \gtrsim \tfrac{1}{4}\, \lambda\!\!\!^{-}_B{}^2 \quad . \tag{17e}$$

The time derivative $\bar{\mathbf{v}}$ of the mean position operator $\bar{\mathbf{r}}$ is the velocity operator one would expect based on the correspondence principle and classical relativistic kinematics. Its components $\bar{v}_j$, $\bar{v}_k$ commute, $[\bar{v}_j, \bar{v}_k] = 0$. Furthermore, $\bar{\mathbf{v}}$ commutes with H_D, i.e., it is a constant of the motion.

Equation (5) is motivated by the requirement that it leaves the subspaces $\mathcal{H}_\pm$ separately invariant. However, this requirement is not sufficient for a unique definition of a relativistic position operator. The Dirac equation becomes diagonal in the Foldy-Wouthuysen (FW) representation.[30] If we require the mean position operator to be diagonal in this representation, it can be obtained in the standard representation via an inverse FW transform:[3]

$$\bar{\mathbf{r}}_{\text{NW}}(t) = \mathbf{r} + \bar{\mathbf{v}} t - \frac{\hbar\beta}{2i\Lambda_D}\left[c\boldsymbol{\alpha} - \frac{c^3 \left(\boldsymbol{\alpha}\cdot\mathbf{p}\right)\mathbf{p}}{\Lambda_D(\Lambda_D + mc^2)}\right] - \frac{c^2\, \mathbf{S}\times\mathbf{p}}{\Lambda_D(\Lambda_D + mc^2)} \quad . \tag{18}$$

This operator is often called the Newton-Wigner position operator.[31] In contrast with the position operator $\bar{\mathbf{r}}$ in Eq. (5), the components of $\bar{\mathbf{r}}_{\text{NW}}$ *do* commute. We see that $\bar{\mathbf{r}}_{\text{NW}}(t)$ shares with $\bar{\mathbf{r}}(t)$ from Eq. (5) that the time derivative $\bar{\mathbf{v}}(t)$ is given by Eq. (1a), i.e., it does not show any *zitterbewegung* because it is a constant of the motion.

General requirements for any position observable describing the localization of a particle or wave packet are discussed in Ref. 3. In this context, the operator $\bar{\mathbf{r}}$ in Eq. (5) appears inappropriate because its components do not commute. The optimal choice for a position observable is the operator $\bar{\mathbf{r}}_{\text{NW}}$. However, general arguments prohibit the possibility of strict spatial localization for a one-particle state (see, e.g., Refs. 3,32,33). This imposes restrictions on the utility of any position operator in relativistic systems.

III. RASHBA (AND PAULI) HAMILTONIAN

An intriguing example of *zitterbewegung*-like dynamics exhibited by a non-Dirac-like Hamiltonian has been found[17] in the Rashba model.[20] This model describes 2D electrons in semiconductor heterostructures with spin-orbit coupling present, using the effective Hamiltonian (we assume here $B = 0$, see Sec. VII for the case $B \neq 0$)

$$H = \frac{p_x^2 + p_y^2}{2m} + H_R \quad . \tag{19a}$$

Here

$$H_R = \alpha\, (\boldsymbol{\sigma} \times \mathbf{p}) \cdot \mathbf{e}_z = \alpha \begin{pmatrix} 0 & p_y + i p_x \\ p_y - i p_x & 0 \end{pmatrix} \tag{19b}$$

is the Rashba term with Rashba coefficient α (Ref. 20), and $\mathbf{e}_z$ denotes the unit vector in the direction perpendicular to the 2D plane. (Note that $H_R^2 = \alpha^2 p^2$, similar

to the Dirac Hamiltonian.) The Hamiltonian (19) is also equivalent to the Pauli Hamiltonian[2] for a 2D system. The energy eigenvalues of H are

$$E_\pm(\mathbf{p}) = \frac{p^2}{2m} \pm \alpha p \quad . \tag{20}$$

The time-dependent position operator $\mathbf{r}(t)$ in the Rashba model was discussed previously in Ref. 17. Evaluated in close analogy to the Dirac case, it is again possible to decompose $\mathbf{r}(t)$ into a mean part $\bar{\mathbf{r}}(t)$ and an oscillating part $\tilde{\mathbf{r}}(t)$. The result is of the form shown in Eq. (2), where $\mathbf{F}$ and $\hat{\omega}(\mathbf{p})$ are now

$$\mathbf{F} = \frac{\partial H_R}{\partial \mathbf{p}} - \frac{\alpha^2 \mathbf{p}}{H_R} = \sigma_z \, \frac{\alpha^2 \, \mathbf{e}_z \times \mathbf{p}}{i H_R} \; , \qquad \hat{\omega}(\mathbf{p}) = \frac{2 H_R}{\hbar} \; . \tag{21}$$

Explicit evaluation shows that $\mathbf{r}(t)$ oscillates with the frequency $\omega_R = 2\alpha p/\hbar$, which is equal to the precession frequency of a spin moving in the effective magnetic field of the Rashba term [see Eq. (24b) below]. The oscillation becomes arbitrarily slow for $p \to 0$. We find for the oscillating part of $\mathbf{r}(t)$

$$\tilde{r}^2(t) = \lambda\!\!\!^{-}_B{}^2/4 \quad , \tag{22}$$

i.e., the magnitude of the oscillations is of the order of the de Broglie wave length $\lambda\!\!\!^{-}_B$ and independent of the Rashba coefficient α. Note that $\lambda\!\!\!^{-}_B$ diverges in the limit $p \to 0$.

We obtain the mean part $\bar{\mathbf{r}}(t)$ by projecting on the subspaces of H associated with the spin-split bands, as in Eq. (5). We find the same $\bar{\mathbf{r}}(t)$ by applying an inverse FW transformation, similar to Eq. (18). For the Rashba model, the last term in Eq. (2a) corresponds to a spatial separation of up and down spin contributions in a wave packet by $\sim \lambda\!\!\!^{-}_B$ (independent of the Rashba coefficient α), which was noticed in previous numerical work.[34] The general validity of Eq. (2a) for two-band models implies the existence of similar displacements for the Dirac and Luttinger cases. See also Ref. 18. The components $\bar{x}$ and $\bar{y}$ of the mean position operator $\bar{\mathbf{r}}$ commute, similar to $\bar{\mathbf{r}}_{\mathrm{NW}}$ in Eq. (18).

The velocity operator and its derivative are given by Eqs. (1) and (3), respectively, using expressions (21). The oscillatory part of $\mathbf{v}$ satisfies

$$\tilde{v}^2(t) = \alpha^2 \quad , \tag{23a}$$

i.e., the magnitude of the oscillatory motion $\tilde{\mathbf{v}}(t)$ is given by the Rashba coefficient α. On the other hand, the components of $\mathbf{v}$ do not commute, and we have

$$[v_x(t), v_y(t)] = 2i\alpha^2 \, \sigma_z \, \mathrm{e}^{-i\hat{\omega}(\mathbf{p})t} \quad , \tag{23b}$$

which implies

$$\Delta v_x \, \Delta v_y \geq \alpha^2 = (\tfrac{1}{2}\omega_R \lambda\!\!\!^{-}_B)^2 \quad , \tag{23c}$$

analogous to Eqs. (15). In Eq. (23c) we replaced the matrix-valued RHS of Eq. (23b) by the eigenvalues of this matrix. Thus similar to the Dirac case, both the magnitude of the oscillations in $\mathbf{v}(t)$ and the minimum uncertainty are given by the same parameter. The components of the mean part of the velocity operator commute, $[\bar{v}_x(t), \bar{v}_y(t)] = 0$. They also commute with H_R, i.e., they are constants of the motion.

The time dependence of orbital angular momentum L_z, spin component S_z, and total angular momentum $J_z = L_z + S_z$ can be straightforwardly discussed. We get

$$L_z(t) \;=\; L_z + \frac{\hbar\,\sigma_z}{2} \left(1 - \mathrm{e}^{-i\hat{\omega}(\mathbf{p})t} \right) , \tag{24a}$$

$$S_z(t) \;=\; \frac{\hbar\,\sigma_z}{2} \, \mathrm{e}^{-i\hat{\omega}(\mathbf{p})t} \; , \tag{24b}$$

$$J_z(t) \;=\; J_z = L_z + S_z \quad . \tag{24c}$$

The formal structure of these equations is analogous to the Dirac-case counterparts shown in Eqs. (13). Equation (24b) represents the well-known spin precession in the effective magnetic field of the Rashba term, which has been observed experimentally.[25] The total angular momentum component perpendicular to the plane does not depend on time, as expected from $[J_z, H] = 0$. Obviously Eqs. (24) require that the spin precession is caused by the *effective* in-plane magnetic field of a spin-orbit coupling term such as the Rashba term. We see here clearly the difference between spin precession caused by spin-orbit coupling and spin precession caused by the Zeeman term in the presence of an external in-plane magnetic field. In the latter case J_z is not a constant of motion.

IV. LUTTINGER HAMILTONIAN

The uppermost valence band Γ_8^v of common semiconductors like Ge and GaAs is well-characterized by the Luttinger Hamiltonian[21]

$$H = -\frac{\gamma_1 \, p^2}{2m} + H_L \quad . \tag{25a}$$

We assume $B = 0$ and use the spherical approximation[35]

$$H_L = \frac{\bar{\gamma}}{m} \left[(\mathbf{p} \cdot \mathbf{S})^2 - \tfrac{5}{4} \, p^2 \mathbb{1}_{4\times 4} \right] \quad , \tag{25b}$$

where γ_1 and $\bar{\gamma}$ are the dimensionless Luttinger parameters, and $\mathbf{S}$ is the vector of 4×4 spin matrices for a system with spin $s = 3/2$. (Note $H_L^2 = \bar{\gamma}^2 p^4/m^2$, similar to the Dirac Hamiltonian.) The twofold-degenerate energy eigenvalues of H are

$$E_\pm(\mathbf{p}) = -\frac{p^2}{2m} \, (\gamma_1 \pm 2\bar{\gamma}) \quad . \tag{26}$$

The upper sign corresponds to the so-called light-hole (LH) states with spin-z component $M = \pm 1/2$, and the lower sign corresponds to the heavy-hole (HH) states with $M = \pm 3/2$. The momentum-dependent energy gap between HH and LH states is

$$\hbar\omega_L = 2\bar{\gamma} p^2/m \quad . \tag{27}$$

The position operator is of the form shown in Eq. (2) with $\mathbf{F}$ and $\hat{\omega}(\mathbf{p})$ given by

$$\mathbf{F} = \frac{\partial H_L}{\partial \mathbf{p}} - \frac{2\mathbf{p}\, H_L}{p^2}\ , \qquad \hat{\omega}(\mathbf{p}) = \frac{2H_L}{\hbar} \quad . \tag{28}$$

Thus $\mathbf{r}(t)$ oscillates with the frequency ω_L, which has been noticed in previous numerical work.[36] Similar to the Rashba Hamiltonian, these oscillations become arbitrarily slow for $p \to 0$. The squared amplitude of the oscillations of $\mathbf{r}(t)$ is $\tilde{r}^2(t) = (3/2)\lambdabar_B^2$, independent of the Luttinger parameter $\bar{\gamma}$. It diverges for $p \to 0$.

We obtain the mean position operator $\bar{\mathbf{r}}$, defined in Eq. (5), using projection operators that project onto HH and LH states.[38] The result coincides with Eq. (2a). The components of $\bar{\mathbf{r}}$ do not commute,

$$[\bar{r}_j, \bar{r}_k] = \left[\frac{\hbar\, \partial H_L/\partial p_j}{2\, H_L}, \frac{\hbar\, \partial H_L/\partial p_k}{2\, H_L}\right] \quad , \tag{29}$$

implying the uncertainty relation

$$\Delta \bar{r}_j\, \Delta \bar{r}_k \geq \frac{3\, \varepsilon_{jkl}\, \hbar^2 p_l}{4\, p^3} \quad . \tag{30}$$

This uncertainty is of the order of (or less than) the de Broglie wave length. The uncertainty is the largest for those components $\bar{r}_j$ that are perpendicular to $\mathbf{p}$.

Using Eqs. (28), the velocity operator can be written in the form shown in Eq. (1). For its oscillating part, we find $\tilde{v}^2(t) = 6\bar{\gamma}^2 (p/m)^2$. The components of $\mathbf{v}$ do not commute,

$$[v_j, v_k] = \left[\frac{\partial H_L}{\partial p_j}, \frac{\partial H_L}{\partial p_k}\right] \quad , \tag{31a}$$

which corresponds to the uncertainty relation

$$\Delta v_x\, \Delta v_y \geq \frac{\bar{\gamma}^2\, p}{m^2}\, \sqrt{3(4p_x^2 + 4p_y^2 + 3p_z^2)} \tag{31b}$$

and cyclic permutations thereof, i.e., the uncertainty is approximately limited by $3\bar{\gamma}^2\, (p/m)^2 = \frac{3}{4}(\omega_L\, \lambdabar_B)^2$. Thus again, the magnitude of the oscillations of $\mathbf{v}(t)$ and the minimum uncertainty are characterized by the same combination of parameters. The velocity $\mathbf{v}(t)$ is not a conserved quantity but satisfies Eq. (3). However, the mean velocity operator is again given by Eq. (1a). Its components commute and are constants of the motion.

The time dependence of orbital angular momentum $\mathbf{L}$, spin $\mathbf{S}$, and total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ turns out to be given by Eqs. (13). Note that the time dependence of $\mathbf{S}(t)$ in the Luttinger model corresponds to a spin precession in the absence of any external or effective magnetic field.[37] Again, the total angular momentum does not depend on time, which reflects the fact that $[\mathbf{J}, H] = 0$.

We remark that a similar analysis as presented in this Section also applies to models that neglect the spin degree of freedom. An example is the 3×3 Shockley Hamiltonian that describes spinless holes in the uppermost valence band Γ_5^v of semiconductors like Si.[39,40] Indeed, this is consistent with the fact that *zitterbewegung* for the Dirac case can be studied already in a model with only one spatial dimension, where the Dirac Hamiltonian H_D becomes a 2×2 matrix that reflects the occurence of both signs of the energy in the spectrum of H_D; but this Hamiltonian does not describe the spin degree of freedom.[29] A spin with spin-orbit coupling is not a necessary condition for the oscillatory behavior of $\mathbf{r}(t)$ and $\mathbf{v}(t)$ to occur. The most basic ingredient required for *zitterbewegung*-like effects are several bands separated by a (usually momentum-dependent) gap. Often the splitting of these bands can be described by an *effective* spin-orbit coupling.[35,41]

V. KANE HAMILTONIAN

The Kane Hamiltonian[22] is an effective Hamiltonian that captures the important physics of electrons and holes in narrow-gap semiconductors like InSb. We restrict ourselves to the 6×6 Kane model which includes the lowest conduction band Γ_6^c and the uppermost valence band Γ_8^v (3D, $B = 0$), neglecting the split-off valence band Γ_7^v, because that model permits a fully analytical solution. Then we have

$$H_K = \begin{pmatrix} (E_g/2)\, \mathbb{1}_{2\times 2} & \sqrt{3}\mathcal{P}\, \mathbf{T} \cdot \mathbf{p} \\ \sqrt{3}\mathcal{P}\, \mathbf{T}^\dagger \cdot \mathbf{p} & -(E_g/2)\, \mathbb{1}_{4\times 4} \end{pmatrix} \quad . \tag{32}$$

Here E_g is the fundamental energy gap, and $\mathcal{P}$ denotes Kane's momentum matrix element. The vector $\mathbf{T}$ of 2×4 matrices is defined in Ref. 42. The energy eigenvalues of H_K are (each twofold degenerate)

$$E_\pm(\mathbf{p}) = \pm \Lambda_K\, , \qquad E_0(\mathbf{p}) = -E_g/2\, , \tag{33a}$$

where

$$\Lambda_K = \sqrt{(E_g/2)^2 + \tfrac{2}{3}\mathcal{P}^2 p^2} \quad , \tag{33b}$$

i.e., the Kane Hamiltonian combines the gapped spectrum of the Dirac Hamiltonian with the gapless spectrum of the Luttinger Hamiltonian. (Indeed, the Luttinger Hamiltonian corresponds to the limiting case $E_g \to \infty$ of the Kane model.) The energy spectrum $E_{\pm,0}(p)$ is shown in Fig. 1.

Results similar to those discussed below can also be derived perturbatively for the full 8×8 Kane Hamiltonian that includes the split-off valence band Γ_7^v. We also remark that a simplified 4×4 Kane Hamiltonian, which includes only the conduction band Γ_6^c and the valence band Γ_7^v, is strictly equivalent to the Dirac Hamiltonian. Recently, *zitterbewegung* was studied for a simplified version of the Kane model where the HH band [with dispersion $E_0(p) = -E_g/2$] and the split-off band Γ_7^v were neglected.[12] In this limit, the Kane Hamiltonian becomes similar to the Dirac Hamiltonian. Our analysis below shows that qualitatively new aspects arise when the HH band is taken into account.

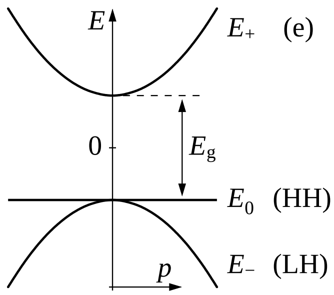

FIG. 1: Energy spectrum $E_{\pm,0}(\mathbf{p})$ of the 6×6 Kane model. Here, E_g denotes the fundamental gap. Each band $E(\mathbf{p})$ is twofold degenerate.

Similar to the Dirac equation, the velocity operator $\mathbf{v}$ in the Kane model has a discrete spectrum. Each component of $\mathbf{v}$ has eigenvalues $\pm\sqrt{2/3}\,\mathcal{P}$ and 0, which correspond to (pure) electron, LH, and HH states. In general, for a wave packet containing a superposition of electron, LH, and HH states we have a finite probability to measure each of these discrete values. The components of the velocity $\mathbf{v}$ do not commute. We get

$$[v_j, v_k] = -\frac{2i}{3}\,\mathcal{P}^2\,\varepsilon_{jkl}\begin{pmatrix}\sigma_l & 0\\ 0 & \Sigma_l\end{pmatrix} \quad , \tag{34a}$$

where Σ_l are the 4×4 spin matrices for spin $s = 3/2$. This corresponds to the uncertainty relation for $j \neq k$

$$\Delta v_j\,\Delta v_k \geq \frac{\mathcal{P}^2}{6} \quad . \tag{34b}$$

Note that Eq. (34a) implies that the minimum uncertainty depends on the dominant character of the wave function. The lower bound $\mathcal{P}^2/6$ requires a LH state. For an electron state, the minimum uncertainty is $\mathcal{P}^2/3$, whereas for a HH state it is $\mathcal{P}^2/2$.

We omit here the lengthy expressions for $\mathbf{r}(t)$ and $\mathbf{v}(t)$. It follows from Eq. (33a) that the oscillating parts $\tilde{\mathbf{r}}(t)$ and $\tilde{\mathbf{v}}(t)$ of $\mathbf{r}(t)$ and $\mathbf{v}(t)$ depend on the frequencies

$$\hbar\omega_{+-} \;\equiv\; E_+ - E_- = 2\Lambda_K \quad , \tag{35a}$$

$$\hbar\omega_{\pm 0} \;\equiv\; E_\pm - E_0 = E_g/2 \pm \Lambda_K \quad . \tag{35b}$$

Unlike in the models discussed above, $\tilde{r}^2(t)$ and $\tilde{v}^2(t)$ in the Kane model are not diagonal in spin space. Hence these quantities depend explicitly on time, oscillating with the frequencies given in Eqs. (35). However, we can estimate the magnitude of these quantities by neglecting the oscillatory terms and diagonalizing the resulting matrices. We get the following twofold-degenerate eigenvalues for $\tilde{r}^2(t)$

$$\tilde{r}^2(t) \simeq \begin{cases} \dfrac{3\,\hbar^2}{2\,p^2} \quad , \\[2ex] \left(\dfrac{7\Lambda_K^4 + E_g^2\Lambda_K^2/4 - E_g^4/8}{8\Lambda_K^4} \pm \dfrac{3\,E_g}{8\Lambda_K}\right)\dfrac{\hbar^2}{p^2} \,. \end{cases} \tag{36}$$

A Taylor expansion shows that for small mean velocities ("nonrelativistic limit") we thus have two characteristic length scales for the oscillatory motion, the de Broglie wave length $\lambda\!\!\!^{-}_B$ and an effective Compton wave length[12]

$$\lambda\!\!\!^{-}_K \equiv \frac{\hbar\mathcal{P}}{E_g} \quad . \tag{37}$$

We have $\lambda\!\!\!^{-}_K \simeq 7$ Å in GaAs and $\lambda\!\!\!^{-}_K \simeq 40$ Å in InSb which should be compared with $\lambda\!\!\!^{-}_D = 3.9 \times 10^{-3}$ Å. Note that, in the nonrelativistic limit, the de Broglie wave length becomes a *fourfold* degenerate eigenvalue of $\tilde{r}^2(t)$, i.e., it characterizes the oscillatory motion of electron, HH, and LH states. For large mean velocities ("relativistic limit"), the de Broglie wave length is the only length scale characterizing $\tilde{r}(t)$. Similarly, we get for $\tilde{v}^2(t)$

$$\tilde{v}^2(t) \simeq \begin{cases} \mathcal{P}^2 \\ \mathcal{P}^2\left(\frac{5}{6}\Lambda_K^2 + \frac{1}{6}E_g^2 \pm \frac{1}{4}E_g\Lambda_K\right)/\Lambda_K^2 \quad , \end{cases} \tag{38}$$

i.e., the magnitude of $\tilde{v}$ is of the order of $\mathcal{P}$ for both small and large mean velocities. Again, the minimum uncertainty of $\mathbf{v}$ [Eq. (34b)] and the magnitude of the oscillations of $\mathbf{v}$ are characterized by the same parameter.

The mean velocity reads

$$\bar{\mathbf{v}}(t) = \left(H_K + \frac{E_g}{2}\right)\left(1 - \frac{H_K E_g}{2\Lambda_K^2}\right)\frac{\mathbf{p}}{p^2} \,. \tag{39}$$

The components of $\bar{\mathbf{v}}$ commute with each other and they are constants of the motion. The mean position operator reads

$$\bar{\mathbf{r}}(t) = \mathbf{r} + \bar{\mathbf{v}}t + \left\{1 - \frac{3E_g\,H_K}{p^2\mathcal{P}^2}, \frac{\hbar^2\,\dot{\mathbf{v}}}{4\Lambda_K^2}\right\} + \left[\frac{\partial \tilde{H}_K^2}{\partial \mathbf{p}} - \frac{2\mathbf{p}\tilde{H}_K^2}{p^2}\right]\frac{\hbar\,[3\Lambda_K^2 + (E_g/2)^2]}{8i\,\tilde{H}_K^2\Lambda_K^2} \,, \tag{40a}$$

where $\{A, B\} = \frac{1}{2}(AB + BA)$ denotes the symmetrized product of A and B,

$$\tilde{H}_K^2 \equiv \left(\frac{E_g^2}{H_K^2} - 2\right)\left(\Lambda_K^2 - \frac{E_g^2}{2}\right) \quad , \tag{40b}$$

and $\dot{\mathbf{v}}$ denotes the acceleration $\dot{\mathbf{v}} = (i/\hbar)[H_K, \mathbf{v}]$. The components of $\bar{\mathbf{r}}(t)$ do not commute with each other. We do not give here the lengthy expressions.

Orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and spin $\mathbf{S}$ also oscillate as a function of time. Similar to the Dirac and Luttinger cases, these oscillations arise even though free particles are considered with no external or effective magnetic field present. However, the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ does not oscillate as a function of time which, as always, reflects the fact that $[\mathbf{J}, H_K] = 0$.

VI. LANDAU HAMILTONIAN

There are several remarkable similarities between the spin-dependent Hamiltonians discussed above and the well-known and rather simple case of the Landau Hamiltonian[26] describing the cyclotron motion of 2D electrons in the presence of a magnetic field $B_z > 0$ perpendicular to the 2D plane. The Landau Hamiltonian is given by

$$H_c = \frac{p_x^2 + p_y^2}{2m} \quad , \tag{41}$$

where $\mathbf{p}$ is the kinetic momentum with

$$[p_x, p_y] = -i\,\hbar\, eB_z \quad . \tag{42}$$

For the elementary charge e we use the convention $e = |e|$. The time-dependent position operator

$$\mathbf{r}(t) = \mathbf{r} + \frac{\mathbf{p}}{m\omega_c}\sin(\omega_c t) + \frac{\mathbf{p}\times\mathbf{e}_z}{m\omega_c}\left[\cos(\omega_c t) - 1\right] \tag{43a}$$

can be written in a compact form using the complex notation $R = x - iy$ and $P = p_x - ip_y$ (Ref. 43), which highlights the analogies between the Landau Hamiltonian and the models in the preceding sections. We get:

$$R(t) = R + \frac{P}{m}\,\frac{1 - \mathrm{e}^{-i\omega_c t}}{i\omega_c} \quad , \tag{43b}$$

where $\omega_c = eB_z/m$ is the cyclotron frequency. Equation (43b) shows that P/m behaves similar to the $\mathbf{F}$ operators in the preceding sections.[44] The magnitude of the oscillations of $R(t)$ is the radius $\Lambda_c = p/(m\omega_c)$ of the cyclotron orbit. Ignoring the oscillations with frequency ω_c, we have

$$\bar{R}(t) = R - \frac{iP}{m\omega_c} \equiv C \tag{44}$$

independent of t, which corresponds to the center of the cyclotron orbit (the guiding center). The components $\bar{x}$ and $\bar{y}$ of $\bar{R}$ do not commute

$$[\bar{x}, \bar{y}] = i\lambda_c^2 \quad , \tag{45a}$$

where $\lambda_c = \sqrt{\hbar/(eB_z)}$ is the magnetic length. Equation (45a) can be written as an uncertainty relation

$$\Delta\bar{x}\,\Delta\bar{y} \geq \tfrac{1}{2}\,\lambda_c^2 \quad . \tag{45b}$$

The velocity operator, in complex notation $V = v_x - iv_y$, is given by

$$V(t) = \frac{P}{m}\,\mathrm{e}^{-i\omega_c t} \quad , \tag{46}$$

so that $\tilde{\mathbf{v}}(t) = \mathbf{v}(t)$ and $\tilde{v}^2(t) = (\omega_c\Lambda_c)^2$. The components v_x and v_y do not commute,

$$[v_x, v_y] = [v_x(t), v_y(t)] = -\frac{i\,\hbar\, eB_z}{m^2} \quad , \tag{47a}$$

which corresponds to the uncertainty relation

$$\Delta v_x\,\Delta v_y \geq \tfrac{1}{2}\,(\omega_c\,\lambda_c)^2 \quad , \tag{47b}$$

which should be compared with Eqs. (15) and (23). Obviously, implications arising from this uncertainty relation become relevant only for sufficiently large magnetic fields when λ_c becomes comparable to Λ_c.

The velocity $V(t)$ is not a conserved quantity, which reflects the effect of the Lorentz force. We have

$$\dot{V}(t) = \frac{-i\omega_c P}{m}\,\mathrm{e}^{-i\omega_c t} \quad . \tag{48}$$

The mean velocity operator vanishes,

$$\bar{V} = 0 \quad , \tag{49}$$

because, on average, the particle is at rest for $B_z \neq 0$. This also implies $[\bar{v}_x, \bar{v}_y] = 0$.

Our analysis indicates that the dynamical properties of the Landau model bear strong resemblances to those exhibited by models showing *zitterbewegung*-like motion.

VII. LANDAU-RASHBA HAMILTONIAN

An interesting example combining two types of oscillatory motion can be found by considering the interplay between 2D cyclotron motion (Sec. VI) and Rashba spin splitting (Sec. III). The Hamiltonian for that situation reads

$$H_{cR} = H_c + H_R + \frac{g}{2}\,\mu_\mathrm{B}\sigma_z B_z \quad . \tag{50}$$

Here we have also included a Zeeman term with Landé factor g and Bohr magnetic moment $\mu_\mathrm{B} = e\hbar/(2m_e)$ (where m_e denotes the electron mass in vacuum), and the terms H_R and H_c are given in Eqs. (19b) and (41). For the following calculation we replace the components p_x and p_y of the kinetic momentum by creation and annihilation operation operators for Landau levels, $a^\dagger$ and a, defined in the usual way,

$$a = \frac{\lambda_c\,P}{\sqrt{2}\hbar} \quad , \tag{51}$$

and $a^\dagger$ is the adjoint of a. The resulting expression for H_{cR} (Ref. 27) is equivalent to the Jaynes-Cummings

model[45] in the rotating-wave approximation. To find the time evolution of the observables in the Heisenberg picture, we first separate H_{cR} into two commuting parts, $H_{cR} = H^{(1)}_{cR} + H^{(2)}_{cR}$, where

$$H^{(1)}_{cR} = \hbar\omega_c \left(a^\dagger a + \frac{1+\sigma_z}{2} \right) \quad , \tag{52a}$$

$$H^{(2)}_{cR} = \frac{\sqrt{2} i\hbar\,\alpha}{\lambda_c} \left(a\sigma_+ - a^\dagger \sigma_- \right) - \frac{\hbar\omega_c}{2} \left(1 - \frac{gm}{2m_\mathrm{e}} \right) \sigma_z \quad . \tag{52b}$$

Here we used $\sigma_\pm \equiv (\sigma_x \pm i\sigma_y)/2$.

It is straightforward to calculate the time evolution of the spin component parallel to the magnetic field,

$$S_z(t) = S_z - i\sigma_z H_R \, \frac{1 - \mathrm{e}^{-2iH^{(2)}_{cR} t/\hbar}}{2iH^{(2)}_{cR}/\hbar} \quad . \tag{53}$$

This result is the generalization of Eq. (24b) to the case of a finite magnetic field. Interestingly, time averaging the r.h.s of Eq. (53) does not result in a vanishing spin component parallel to the field direction. We find

$$\bar{S}_z = -\frac{\hbar^2 \omega_c}{4\, H^{(2)}_{cR}} \left(1 - \frac{gm}{2m_\mathrm{e}} \right) \quad . \tag{54a}$$

Neglecting Zeeman splitting and considering the limit of small B_z, this result becomes

$$\bar{S}_z \approx -\frac{\hbar^2 \omega_c}{4\alpha p^2} \left(\boldsymbol{\sigma} \times \mathbf{p} \right) \cdot \mathbf{e}_z \quad , \tag{54b}$$

which is exactly the finite value of the spin component parallel to the magnetic field that was obtained in semiclassical calculations of spin-split cyclotron orbits.[46]

To calculate the time evolution of the position operator, we use the complex notation from Sec. VI. We have

$$R = C + \frac{iP}{m\omega_c} \equiv C + i\sqrt{2}\lambda_c\, a \quad , \tag{55}$$

where C is the position of the guiding center, see Eq. (44). Even in the presence of H_R, the guiding center C remains a constant of the motion, $[C, H_{cR}] = 0$. The time evolution of P due to $H^{(1)}_{cR}$ is just a trivial factor $\mathrm{e}^{-i\omega_c t}$, so that we only need to evaluate the time evolution of $P \propto a$ under $H^{(2)}_{cR}$. (Note that $[H^{(1)}_{cR}, H^{(2)}_{cR}] = 0$.) This problem has been solved for the Jaynes-Cummings model.[45,47] Translating into our situation, we get for the time-dependent position operator

$$R(t) = C + \frac{i \exp\left[-i(\omega_c + \omega_+)t\right]}{\omega_- - \omega_+} \left(\frac{\omega_-}{\omega_c} \frac{P}{m} + 2i\alpha\sigma_- \right) - \frac{i \exp\left[-i(\omega_c + \omega_-)t\right]}{\omega_- - \omega_+} \left(\frac{\omega_+}{\omega_c} \frac{P}{m} + 2i\alpha\sigma_- \right) \quad , \tag{56a}$$

with the frequency operators $\omega_\pm$ given by

$$\hbar\omega_\pm = -H^{(2)}_{cR} \pm \sqrt{\left(H^{(2)}_{cR} \right)^2 + 2\hbar\omega_c m\alpha^2} \quad . \tag{56b}$$

The terms proportional to σ_- in Eq. (56a) are reminiscent of the oscillatory motion in the Rashba case for $B_z = 0$, where the amplitude of the oscillations is inversely proportional to the de Broglie wave length and independent of α, see Eq. (22). Here these terms contribute to a spin-dependent renormalization of the cyclotron radius. We also note that $\bar{R}(t) = C$, so that Eq. (45) remains valid in the presence of H_R.

The velocity operator is given by

$$V \equiv \dot{R} = \frac{P}{m} - 2i\alpha\sigma_- \quad . \tag{57}$$

The commutator of the components of V,

$$[v_x, v_y] = -\frac{i\hbar\, eB_z}{m^2} + 2i\alpha^2 \sigma_z \quad , \tag{58a}$$

is the sum of the corresponding results obtained separately from H_c and H_R [see Eqs. (23b) and (47a)]. However, in the uncertainty relation

$$\Delta v_x \, \Delta v_y \geq \left| \frac{\hbar\, eB_z}{2m^2} - \alpha^2 \right| , \tag{58b}$$

the two contributions are subtracted, thus *reducing* the minimum uncertainty. The time dependence of V can be readily obtained by taking the time derivative of Eq. (56a). It can be written as

$$V(t) = \mathrm{e}^{-i(\omega_c + \omega_+)t} F_+ + \mathrm{e}^{-i(\omega_c + \omega_-)t} F_- \quad , \tag{59a}$$

with the complex amplitude operators

$$F_\pm = \frac{V}{2} \pm \left(\frac{H^{(2)}_{cR} + 2m\alpha^2}{\omega_+ - \omega_-} \frac{P}{m} + 2i\alpha \, \frac{H^{(2)}_{cR} - \hbar\omega_c}{\omega_+ - \omega_-} \, \sigma_- \right) . \tag{59b}$$

VIII. CONCLUSIONS AND OUTLOOK

We studied a variety of qualitatively different model Hamiltonians for quasi-free electrons that exhibit *zitterbewegung*-like oscillatory motion. A number of features

can be identified that are widely shared as discussed in Sec. I. Here we finally point out open questions.

For the Dirac Hamiltonian, the amplitude of the *zitterbewegung* of $\mathbf{r}(t)$ is given by the Compton wavelength in the nonrelativistic limit and by the de Broglie wave length in the relativistic limit. For those Hamiltonians having a gap that vanishes for $p \to 0$, the length scale of oscillations in $\mathbf{r}(t)$ is always given by the de Broglie wavelength $\lambda\!\!\!^{-}_B = \hbar/p$, independent of the magnitude of spin-orbit coupling. It is surprising that the amplitude of the oscillations of $\mathbf{r}(t)$ diverges in the nonrelativistic limit $p \to 0$.

The most interesting but also, at least in our present work, a largely open aspect is the experimental observability of *zitterbewegung*-like effects. Certainly, any measurement of the oscillatory motion must obey the fundamental uncertainty relations [Eq. (4)] discussed in our work. Furthermore, we have already commented on the intimate relation between oscillations in position and spin space. However, while spin precession due to spin-orbit coupling can be observed experimentally,[25] it is often argued that the *zitterbewegung* of $\mathbf{r}(t)$ "is not an observable motion, for any attempt to determine the position of the electron to better than a Compton wavelength must defeat its purpose by the creation of electron-positron pairs" (Ref. 5). We note that the same argument can be applied to Bloch electrons in solids where electron-hole pairs can be created.[16]

Acknowledgments

RW thanks D. Culcer and UZ thanks P. Schwerdtfeger and G. Vignale for enlightening discussions. UZ is supported by the Marsden Fund Council from Government funding, administered by the Royal Society of New Zealand. Research at KITP was supported by the National Science Foundation under Grant No. PHY99-07949.

APPENDIX: IMPORTANT FORMULAE

Here we briefly summarize important formulae that are used in our discussion of the oscillatory motion in various models. The Heisenberg equation of motion for an operator A reads

$$\frac{dA}{dt} = \frac{i}{\hbar}[H, A] \quad . \tag{A.1}$$

It has the formal solution

$$A(t) = \mathrm{e}^{iHt/\hbar}\, A(0)\, \mathrm{e}^{-iHt/\hbar} \quad . \tag{A.2}$$

In particular, the velocity operator $\mathbf{v}$ is defined by the Heisenberg equation of motion for the position operator $\mathbf{r}$,

$$\mathbf{v} \equiv \frac{d\mathbf{r}}{dt} = \frac{i}{\hbar}[H, \mathbf{r}] \quad . \tag{A.3}$$

Throughout we use the convention that $A(t)$ denotes an operator in the Heisenberg picture and $A = A(0)$ is the corresponding operator in the Schrödinger picture.

In general, the uncertainty principle for two noncommuting observables A and B reads

$$\Delta A\, \Delta B \geq \tfrac{1}{2}\, |\langle [A, B] \rangle| \quad , \tag{A.4}$$

where the uncertainty ΔA of A is defined as

$$\Delta A \equiv \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \quad . \tag{A.5}$$

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[44] This is only a partial analogy, because P/m acts as a unidirectional ladder operator on the infinite set of Landau levels [see Eq. (51)], while the **F** operators map states between two bands. However, note also that an alternative sign convention for the complex notation is $P' = p_x + ip_y$ so that $P' \propto a^\dagger$.

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